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On the characteristic and Laplacian polynomials of trees

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We find the characteristic polynomials of adjacency and Laplacian matrices of arbitrary unweighted rooted trees in term of vertex degrees, using the concept of the rooted product of graphs. Our result generalizes a result of Rojo and Soto [O. Rojo, R. Soto, The spectra of the adjacency matrix and Laplacian matrix for some balanced trees, Linear Algebra Appl. 403 (2005) 97–117] on a special class of rooted unweighted trees, namely the trees such that their vertices in the same level have equal degrees.

1. Introduction and results

Computing characteristic polynomials of adjacency and Laplacian matrix of graphs is one of the essential topics in the theory of graphs. Trees, which are acyclic connected graphs, are widely used in various fields of sciences. A well known formula for the characteristic polynomial of a tree with n vertices is $\sum_i c_i\lambda^{n-i}$, where the odd coefficients are zero, and the even coefficients are given by the rule that $(-1)^r c_{2r}$ is the number of ways of choosing r disjoint edges in the tree (see [1, p. 49]). For a rooted tree of three levels there is an exact formula for the characteristic polynomial of its adjacency matrix in terms of vertex degrees (see for example [10]).

Let G be a simple graph. Throughout the paper $A(G)$, $D(G)$, and $L(G) = D(G) - A(G)$, denote the adjacency matrix, the diagonal matrix of vertex degrees, and the Laplacian matrix of G, respectively. The characteristic polynomials of $A(G)$ and $L(G)$ are denoted by $P_G(\lambda)$ and $Q_G(\lambda)$, respectively. Finally we denote the spectrum of a matrix M by $\sigma(M)$.

Let x_1 be a vertex of degree 1 in the graph G and let x_2 be the vertex adjacent to x_1. Let G_1 be the induced subgraph obtained from G by deleting the vertex x_1 and G_2 be the induced subgraph obtained from G by deleting the vertices x_1 and x_2. Then (see [2, p. 59])

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$$P_G(\lambda) = \lambda P_{G_1}(\lambda) - P_{G_2}(\lambda).$$

By iterating the above formula, the characteristic polynomial of a tree can easily be determined.

Recently computing characteristic polynomials of adjacency and Laplacian matrices of some classes of trees has been the object of many papers (see for example [5–9]). Rojo and Soto in [6] obtained characteristic polynomials of adjacency and Laplacian matrices of a special kind of rooted unweighted trees, namely the trees such that their vertices in the same level have equal degrees.

In this paper we use the concept of rooted product of graphs and find a recursive formula for characteristic polynomials of adjacency and Laplacian matrices of arbitrary rooted unweighted trees, in terms of vertex degrees. Let us recall rooted product of graphs. Suppose that $G = \{G_1, G_2, \ldots, G_k\}$ is a sequence of k rooted graphs and H is a labelled graph on k vertices. The rooted product of H by G, which is denoted by $H(G)$, is obtained by identifying the root vertex of G_i with the ith vertex of H (see [4]). In [3] the characteristic polynomials of $A(H(G))$ and $L(H(G))$ are computed, in terms of the characteristic polynomials of the graphs H and $G_i, i = 1, 2, \ldots, k$.

One can represent a rooted tree in terms of a suitable rooted product. In fact let T be a rooted tree of $k+1$ levels and $G_1, G_2, \ldots, G_n$ be rooted trees of k levels that are obtained by deletion the root vertex of T. Then $T = S_{n+1}(S_1, G_1, G_2, \ldots, G_n)$ is the rooted product of S_{n+1} by $\{S_1, G_1, G_2, \ldots, G_n\}$, where S_m is the star on m vertices (see Fig. 1).

In order to state the main results of the paper we need some notation. Let T be a rooted tree of k levels. Suppose that n_{k-j+1}, where $j = 1, 2, \ldots, k$, is the number of vertices on level j and $d_{k-j+1,i}$, where $i = 1, 2, \ldots, n_{k-j+1}$, is the degree of the ith vertex of level j. If $e_{j,i}$ denotes the number of neighbors on level $k-j+2$ of the ith vertex from the level $k-j+1$ of T, where $2 \leqslant j \leqslant k$, then

$$e_{j,i} = \begin{cases} d_{j,i} & \text{if } j = k, \\ d_{j,i} - 1 & \text{if } j \neq k. \end{cases}$$

For $j = 1, 2, 3, \ldots, k$, put $m_{j,0} = 0$ and for $1 \leqslant r \leqslant n_{k-j+1}$ put $m_{j,r} = e_{k-j+1,1} + \cdots + e_{k-j+1,r}$. In the following theorem, we find a recursive formula for the characteristic polynomial of the adjacency matrix of a rooted tree of k levels.

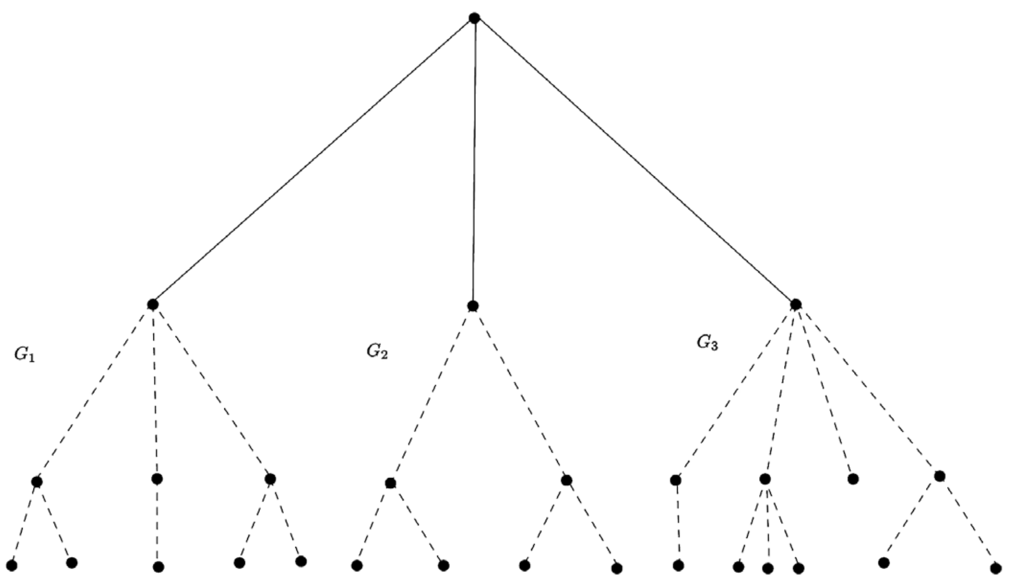

Fig. 1. $S_4(S_1, G_1, G_2, G_3)$, where S_{n+1} is the star on n vertices and G_1, G_2, G_3 are arbitrary rooted trees.

Theorem 1. *Let T be a rooted tree of k levels. Suppose that $P_{0,i} = 1$, where $i = 1, 2, \ldots, n_1$, and $P_{1,i} = \lambda$, where $i = 1, 2, \ldots, n_2$. If $e_{j+1,r} = 0$ we put $P_{j+1,r} = \lambda$, where $j = 1, 2, \ldots, k-1$ and $1 \leqslant r \leqslant n_j$. In other cases we put*

$$P_{j+1,r} = \prod_{i=1+m_{j+1,r-1}}^{m_{j+1,r}} P_{j,i} \left(\lambda - \sum_{i=1+m_{j+1,r-1}}^{m_{j+1,r}} \frac{\prod_{s=1+m_{j,i-1}}^{m_{j,i}} P_{j-1,s}}{P_{j,i}} \right).$$

Then $P_T(\lambda) = P_{k,1}$.

Using a similar argument as in the proof of Theorem 1, we can derive a recursive formula for Laplacian polynomial of a rooted tree.

Theorem 2. *Let T be a rooted tree of k levels. Suppose that $Q_{0,i} = 1$, where $i = 1, 2, \ldots, n_1$ and $Q_{1,i} = \lambda - 1$, where $i = 1, 2, \ldots, n_2$. If $e_{j+1,r} = 0$ we put $Q_{j+1,r} = \lambda - 1$, where $j = 1, 2, \ldots, k-1$ and $1 \leqslant r \leqslant n_j$. In other cases we put*

$$Q_{j+1,r} = \prod_{i=1+m_{j+1,r-1}}^{m_{j+1,r}} Q_{j,i} \left(\lambda - d_{j+1,r} - \sum_{i=1+m_{j+1,r-1}}^{m_{j+1,r}} \frac{\prod_{s=1+m_{j,i-1}}^{m_{j,i}} Q_{j-1,s}}{Q_{j,i}} \right).$$

Then $Q_T(\lambda) = Q_{k,1}$.

2. Proofs

In this section we prove Theorem 1. Also we state some corollaries and examples of Theorems 1 and 2. Let $G = \{G_1, G_2, \ldots, G_k\}$ be a sequence of k rooted graphs of order $n_1, n_2, \ldots, n_k$ respectively, and H be a labelled graph on k vertices. We shall use a suitable labelling of the vertices of $H(G_1, G_2, \ldots, G_k)$ as follows. The root vertex of G_1 has label 1 and the vertices of G_1 have consecutive labels from 1 to n_1. The root vertex of G_2 has label $n_1 + 1$ and the vertices of G_2 have consecutive labels from $n_1 + 1$ to $n_1 + n_2, \ldots$, and finally the root vertex of G_k has label $n_1 + n_2 + \cdots + n_{k-1} + 1$ and the vertices of G_k have consecutive labels from $n_1 + n_2 + \cdots + n_{k-1} + 1$ to $n_1 + n_2 + \cdots + n_k$. Now let $M^{1,1}$ denote the matrix obtained by deletion of the first column and row of a matrix M, and put $\overline{P}_{G_i}(\lambda) = \det(\lambda I - A(G_i)^{1,1})$, $\overline{Q}_{G_i}(\lambda) = \det(\lambda I - L(G_i)^{1,1})$. If G_i is the graph of order 1 put $\overline{P}_{G_i}(\lambda) = \overline{Q}_{G_i}(\lambda) = 1$. We need the following theorem which is proved in [3, Theorem 1] (the first part of Theorem A is proved by Godsil and McKay in [4]):

Theorem A. *Let H and $G_1, G_2, \ldots, G_k, j = 1, 2, \ldots, k$, be simple graphs. If $K = H(G_1, G_2, \ldots, G_k)$ then $P_K(\lambda) = (-1)^k \det(M)$, where*

$$M_{ij} = \begin{cases} -P_{G_i}(\lambda) & \text{if } i = j, \\ \overline{P}_{G_i}(\lambda) & \text{if } i \neq j \text{ and } A(H)_{ij} = 1, \\ 0 & \text{if } i \neq j \text{ and } A(H)_{ij} = 0, \end{cases}$$

$1 \leqslant i, j \leqslant k$, and $Q_K(\lambda) = \det(N)$, where

$$N_{ij} = \begin{cases} Q_{G_i}(\lambda) & \text{if } i = j, \\ \overline{Q}_{G_i}(\lambda) & \text{if } i \neq j \text{ and } A(H)_{ij} = 1, \\ 0 & \text{if } i \neq j \text{ and } A(H)_{ij} = 0, \end{cases}$$

$1 \leqslant i, j \leqslant k$.

We also need the following lemma, which can be proved by an easy induction on n.

Lemma 1. *For $i = 1, 2, \ldots, n$, let x_i be an arbitrary variable. Then*

$$\begin{vmatrix} x_1 & 1 & 1 & \cdots & 1 \\ 1 & x_2 & 0 & \cdots & 0 \\ 1 & 0 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & x_n \end{vmatrix}_{n\times n} = \prod_{i=1}^{n} x_i - \sum_{i=2}^{n} \prod_{j=2, j\neq i}^{n} x_j.$$

Now we are ready to prove a main result of the paper.

Proof of Theorem 1. We prove the assertion by induction on k. First let $k = 2$. Then T is a star on $n_1 + 1$ vertices and

$$P_T(\lambda) = P_{2,1} = \prod_{i=1}^{n_1} P_{1,i}\left(\lambda - \sum_{1}^{n_1} \frac{1}{P_{1,i}}\right) = \lambda^{n_1-1}(\lambda^2 - n_1).$$

Thus the assertion is true for $k = 2$. Suppose that the result is true for all positive integers which are smaller than k. Suppose that T is a rooted tree of k levels. Let $G_1, G_2, \ldots, G_{e_k}$ be rooted trees with root vertex degrees $e_{k-1,1}, e_{k-1,2}, \ldots e_{k-1,e_k}$ that are obtained by deletion the root vertex of T. Then $T = S_{e_k+1}(S_1, G_1, G_2, \ldots, G_{e_k})$. Since

$$A(S_{e_k+1}) = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

using the notation of Theorem A, we have

$$\begin{aligned} M_{11} &= -P_{S_1}(\lambda) = -\lambda, \\ M_{ii} &= -P_{G_{i-1}}(\lambda), \quad 2 \leqslant i \leqslant e_k + 1, \\ M_{1j} &= \overline{P}_{S_1}(\lambda) = 1, \quad 2 \leqslant j \leqslant e_k + 1, \\ M_{i1} &= \overline{P}_{G_{i-1}}(\lambda), \quad 2 \leqslant i \leqslant e_k + 1. \end{aligned}$$

Therefore by Theorem A and Lemma 1 we have

$$P_T(\lambda) = (-1)^{e_k+1} \begin{vmatrix} -\lambda & 1 & 1 & \cdots & 1 \\ \overline{P}_{G_1}(\lambda) & -P_{G_1}(\lambda) & 0 & \cdots & 0 \\ \overline{P}_{G_2}(\lambda) & 0 & -P_{G_2}(\lambda) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{P}_{G_{e_k}}(\lambda) & 0 & 0 & \cdots & -P_{G_{e_k}}(\lambda) \end{vmatrix}$$

$$= (-1)^{e_k+1}\overline{P}_{G_1}(\lambda)\overline{P}_{G_2}(\lambda)\cdots\overline{P}_{G_{e_k}}(\lambda) \begin{vmatrix} -\lambda & 1 & 1 & \cdots & 1 \\ 1 & -\frac{P_{G_1}(\lambda)}{\overline{P}_{G_1}(\lambda)} & 0 & \cdots & 0 \\ 1 & 0 & -\frac{P_{G_2}(\lambda)}{\overline{P}_{G_2}(\lambda)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -\frac{P_{G_{e_k}}(\lambda)}{\overline{P}_{G_{e_k}}(\lambda)} \end{vmatrix}$$

$$= \overline{P}_{G_1}(\lambda)\overline{P}_{G_2}(\lambda)\cdots\overline{P}_{G_{e_k}}(\lambda)\left(\lambda\prod_{i=1}^{e_k}\frac{P_{G_i}(\lambda)}{\overline{P}_{G_i}(\lambda)} - \sum_{i=1}^{e_k}\prod_{j=1,j\neq i}^{e_k}\frac{P_{G_j}(\lambda)}{\overline{P}_{G_j}(\lambda)}\right)$$

$$= \prod_{i=1}^{e_k} P_{G_i}(\lambda)\left(\lambda - \sum_{i=1}^{e_k}\frac{P_{G_i}(\lambda)}{\overline{P}_{G_i}(\lambda)}\right). \tag{1}$$

By the induction hypothesis $P_{G_r}(\lambda) = P_{k-1,r}$, for $1 \leqslant r \leqslant e_k$. Since $\overline{P}_{G_r}(\lambda)$ is the characteristic polynomial of the graph which obtained by deleting the root vertex of G_r, for $1 \leqslant r \leqslant e_k$, we have

$$\overline{P}_{G_r}(\lambda) = \prod_{s=1+m_{k-1,r-1}}^{m_{k-1,r}} P_{k-2,s}.$$

In the special case if $P_{G_r}(\lambda) = \lambda$, using the notation of Theorem *A*, we have $\overline{P}_{G_r}(\lambda) = 1$. Since in the first level of *T* there is one vertex, we have $n_k = 1$, $m_{k,1} = e_k$. Thus by replacing the last results in (1) we obtain that

$$P_T(\lambda) = \prod_{i=1+m_{k,0}}^{m_{k,1}} P_{k-1,i}\left(\lambda - \sum_{i=1+m_{k,0}}^{m_{k,1}}\frac{\prod_{s=1+m_{k-1,i-1}}^{m_{k-1,i}} P_{k-2,s}}{P_{k-1,i}}\right).$$

Therefore $P_T(\lambda) = P_{k,1}$ and the Theorem is proved. □

Taking $k = 3$ in Theorem 1, we can obtain a well known formula, computed in [10], on the characteristic polynomial of the adjacency matrix of a rooted tree of 3 levels. A rooted tree of 3 levels is shown in Fig. 2.

Corollary 1. *Let T be a rooted tree of* 3 *levels and* $n = e_{3,1}$. *Then*

$$P_T(\lambda) = \lambda^{\sum_{i=1}^{n} e_{2,i}-(n-1)}\prod_{i=1}^{n}\left(\lambda^2 - e_{2,i}\right)\left(1 - \sum_{i=1}^{n}\frac{1}{\lambda^2 - e_{2,i}}\right).$$

Proof. Put $k = 3$. By Theorem 1, $P_T(\lambda) = P_{3,1}$. For $i = 1, 2, \ldots, n_2$, $P_{2,i} = \lambda^{e_{2,i}-1}(\lambda^2 - e_{2,i})$ is the characteristic polynomial of the star graph of order $e_{2,i} + 1$ and for $j = 1, 2, \ldots, n_1$, $P_{1,j} = \lambda$. Thus if $n = e_{3,1}$ denotes the degree of root vertex of *T*, then

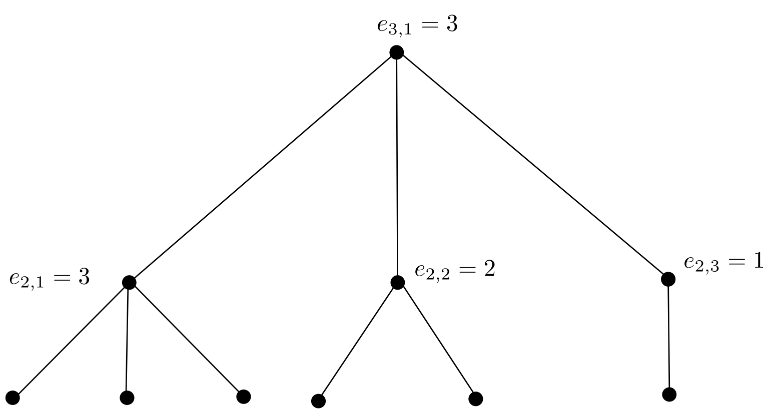

Fig. 2. A rooted tree of 3 levels.

$$
\begin{aligned}
P_T(\lambda) &= \prod_{i=1}^{n} P_{2,i}\left(\lambda - \sum_{i=1}^{n} \frac{\prod_{s=1+m_{2,i-1}}^{m_{2,i}} P_{1,s}}{P_{2,i}}\right) \\
&= \prod_{i=1}^{n} \lambda^{e_{2,i}-1}(\lambda^2 - e_{2,i})\left(\lambda - \sum_{i=1}^{n} \frac{\lambda^{e_{2,i}}}{\lambda^{e_{2,i}-1}(\lambda^2 - e_{2,i})}\right) \\
&= \lambda^{\sum_{i=1}^{n} e_{2,i}-(n-1)} \prod_{i=1}^{n}\left(\lambda^2 - e_{2,i}\right)\left(1 - \sum_{i=1}^{n} \frac{1}{\lambda^2 - e_{2,i}}\right).
\end{aligned}
$$

This completes the proof. □

Now suppose T is a rooted tree of k levels and the vertices of T in the same level have equal degree. In [8] Rojo and Robbiano, called such a tree a generalized Bethe tree. They denoted the class of generalized Bethe trees of k levels by $\mathcal{B}_k$. Rojo and Soto [6] computed the characteristic polynomial of adjacency and Laplacian matrices of graphs in $\mathcal{B}_k$. As a corollary of Theorem 1 we can compute $P_{\beta_k}(\lambda)$, where β_k is a generalized Bethe tree of k levels. If d_{k-j+1} denotes the vertex degree of vertices on level j in β_k, then put

$$
e_j = \begin{cases} d_j & \text{if } j = k, \\ d_j - 1 & \text{if } j \neq k. \end{cases}
$$

Since the vertices on the jth level in β_k have equal degree, $P_{j,1} = P_{j,i}$, for all $i = 2, \ldots, n_{k-j+1}$. If $P_j(\lambda)$ denotes $P_{j,i}$, then by Theorem 1

$$
P_j(\lambda) = P_{j-1}^{e_j-1}(\lambda)\left(\lambda P_{j-1}(\lambda) - e_j P_{j-2}^{e_{j-1}}(\lambda)\right). \tag{2}
$$

Corollary 2. *Let $P_0(\lambda) = 1, P_1(\lambda) = \lambda$ and*

$$
P_j(\lambda) = \lambda P_{j-1}(\lambda) - e_j P_{j-2}(\lambda) \quad \textit{for all } j = 2, 3, \ldots, k.
$$

Then $P_{\beta_k}(\lambda) = P_k(\lambda)\prod_{j=1}^{k-1} P_j^{n_j - n_{j+1}}(\lambda)$.

Proof. The proof is done by using (2) and an easy induction on k. □

Now we state some corollaries of Theorem 2. First we find an exact formula for the characteristic polynomial of the Laplacian matrix of a rooted tree of 3 levels.

Corollary 3. *Let T be a rooted tree of 3 levels and $n = e_{3,1}$. The characteristic polynomial of the Laplacian matrix of T is given by*

$$
Q_T(\lambda) = (\lambda - 1)^{\sum_{i=1}^{n} e_{2,i} - n} \prod_{i=1}^{n}\left(\lambda^2 - (2 + e_{2,i})\lambda + 1\right)\left(\lambda - n - \sum_{i=1}^{n} \frac{\lambda - 1}{\lambda^2 - (2 + e_{2,i})\lambda + 1}\right).
$$

Proof. By Theorem 2, we have to compute $Q_{3,1}$. For $j = 1, 2, \ldots, n_1$, we have $Q_{1,j} = \lambda - 1$ and for $r = 1, 2, \ldots, n_2$ we have

$$
\begin{aligned}
Q_{2,r} &= \prod_{i=1+m_{2,r-1}}^{m_{2,r}} Q_{1,i}\left(\lambda - d_{2,r} - \sum_{i=1+m_{2,r-1}}^{m_{2,r}} \frac{1}{Q_{1,i}}\right) \\
&= \prod_{i=1}^{e_{2,r}} (\lambda - 1)\left(\lambda - e_{2,r} - 1 - \sum_{i=1}^{e_{2,r}} \frac{1}{\lambda - 1}\right) \\
&= (\lambda - 1)^{e_{2,r}-1}\left(\lambda^2 - (2 + e_{2,i})\lambda + 1\right).
\end{aligned}
$$

Now let $n = e_{3,1}$ denotes the degree of root vertex of T. Thus

$$\begin{aligned}
Q_T(\lambda) &= \prod_{i=1}^{n} Q_{2,i}\left(\lambda - n - \sum_{i=1}^{n}\frac{\prod_{s=1+m_{2,i-1}}^{m_{2,i}} Q_{1,s}}{Q_{2,i}}\right)\\
&= \prod_{i=1}^{n}(\lambda-1)^{e_{2,i}-1}\left(\lambda^2-(2+e_{2,i})+1\right)\\
&\quad\times\left(\lambda - n - \sum_{i=1}^{n}\frac{(\lambda-1)^{e_{2,i}}}{(\lambda-1)^{e_{2,i}-1}(\lambda^2-(2+e_{2,i})+1)}\right)\\
&= (\lambda-1)^{\sum_{i=1}^{n} e_{2,i}-n}\prod_{i=1}^{n}\left(\lambda^2-(2+e_{2,i})\lambda+1\right)\left(\lambda-n-\sum_{i=1}^{n}\frac{\lambda-1}{\lambda^2-(2+e_{2,i})\lambda+1}\right).
\end{aligned}$$

So the proof is complete. □

Suppose that β_k is a generalized Bethe tree of k levels. Since the vertices on the jth level in β_k have equal degree, $P_{j,1} = P_{j.i}$, for all $i = 1, 2, \ldots, n_{k-j+1}$. If $P_j(\lambda)$ denotes $P_{j,i}$, then by Theorem 2

$$Q_j(\lambda) = Q_{j-1}^{e_j-1}(\lambda)\left((\lambda-d_j)Q_{j-1} - e_jQ_{j-2}^{e_{j-1}}(\lambda)\right). \tag{3}$$

Using (3) we can compute the characteristic polynomial of the Laplacian matrix of β_k.

Corollary 4. *Let* $Q_0(\lambda) = 1, Q_1(\lambda) = \lambda - 1$ *and*

$$Q_j(\lambda) = (\lambda-d_j)Q_{j-1}(\lambda) - e_jQ_{j-2}(\lambda) \quad \textit{for all } j = 2, 3, \ldots, k.$$

Then $Q_{\beta_k}(\lambda) = Q_k(\lambda)\prod_{j=1}^{k-1} Q_j^{n_j-n_{j+1}}(\lambda)$.

Proof. The proof can be done by using (3) and an easy induction on k. □

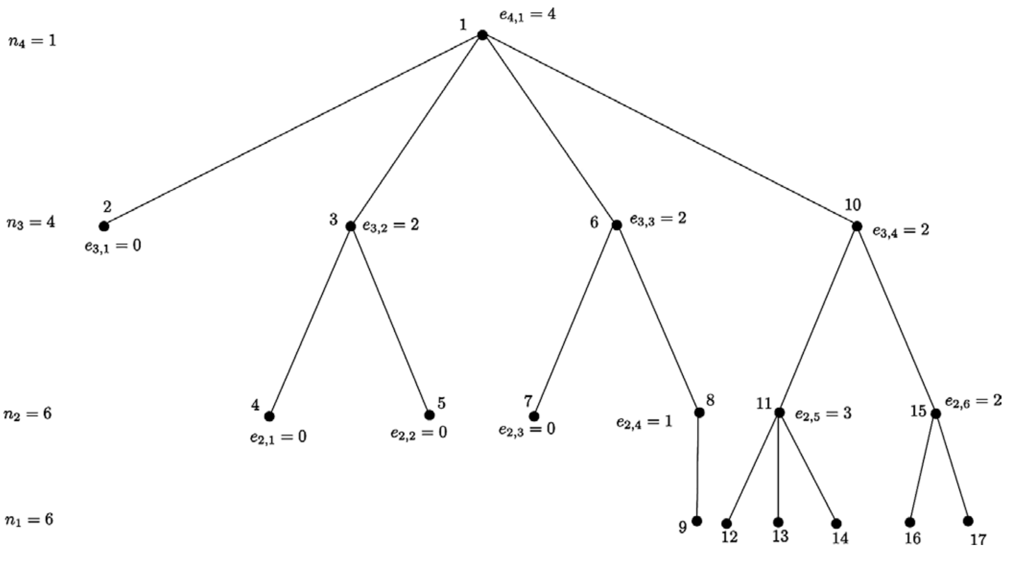

Fig. 3. A rooted tree of 4 levels.

Example 1. Let T be a rooted tree of 4 levels as shown in Fig. 3. In T we have $n_1 = n_2 = 6$, $n_3 = 4$, $n_4 = 1$ and $e_{2,1} = e_{2,2} = e_{2,3} = 0$, $e_{2,4} = 1$, $e_{2,5} = 3$, $e_{2,6} = 2$. The values of $e_{i,j}$ and the labelling of T are shown in Fig. 3.

To compute the characteristic polynomial of the adjacency matrix of T, using Theorem 1, we must calculate $P_{4,1} = P_T(\lambda)$. For vertices on the third level of T we have $P_{2,1} = P_{2,2} = P_{2,3} = \lambda$. By Theorem 1 we have

$$P_{2,4} = P_{1,1}\left(\lambda - \frac{1}{P_{1,1}}\right) = \lambda^2 - 1,$$

$$P_{2,5} = P_{1,2}P_{1,3}P_{1,4}\left(\lambda - \frac{1}{P_{1,2}} - \frac{1}{P_{1,3}} - \frac{1}{P_{1,4}}\right) = \lambda^2(\lambda^2 - 3),$$

$$P_{2,6} = P_{1,5}P_{1,6}\left(\lambda - \frac{1}{P_{1,5}} - \frac{1}{P_{1,6}}\right) = \lambda(\lambda^2 - 2).$$

For vertices on the second level of T we have $P_{3,1} = \lambda$ and

$$P_{3,2} = P_{2,1}P_{2,2}\left(\lambda - \frac{1}{P_{2,1}} - \frac{1}{P_{2,2}}\right) = \lambda(\lambda^2 - 2),$$

$$P_{3,3} = P_{2,3}P_{2,4}\left(\lambda - \frac{1}{P_{2,3}} - \frac{P_{1,1}}{P_{2,4}}\right) = \lambda^4 - 3\lambda^2 + 1,$$

$$P_{3,4} = P_{2,5}P_{2,6}\left(\lambda - \frac{P_{1,2}P_{1,3}P_{1,4}}{P_{2,5}} - \frac{P_{1,5}P_{1,6}}{P_{2,6}}\right) = \lambda^8 - 7\lambda^6 + 11\lambda^4.$$

Therefore $P_T(\lambda)$ can be computed as follows:

$$\begin{aligned} P_{4,1} &= P_{3,1}P_{3,2}P_{3,3}P_{3,4}\left(\lambda - \frac{1}{P_{3,1}} - \frac{P_{2,1}P_{2,2}}{P_{3,2}} - \frac{P_{2,3}P_{2,4}}{P_{3,3}} - \frac{P_{2,5}P_{2,6}}{P_{3,4}}\right) \\ &= \lambda^{17} - 16\lambda^{15} + 95\lambda^{13} - 264\lambda^{11} + 351\lambda^9 - 198\lambda^7 + 34\lambda^5. \end{aligned}$$

To compute the characteristic polynomial of the Laplacian matrix of T, by Theorem 2, we have to calculate $Q_{4,1} = Q_T(\lambda)$. For vertices on the third level of T we have $Q_{2,1} = Q_{2,2} = Q_{2,3} = \lambda - 1$. By Theorem 2 we have

$$Q_{2,4} = Q_{1,1}\left(\lambda - 1 - \frac{1}{Q_{1,1}}\right) = \lambda^2 - 2\lambda,$$

$$Q_{2,5} = Q_{1,2}Q_{1,3}Q_{1,4}\left(\lambda - 3 - \frac{1}{Q_{1,2}} - \frac{1}{Q_{1,3}} - \frac{1}{Q_{1,4}}\right) = \lambda(\lambda - 4)(\lambda - 1)^2,$$

$$Q_{2,6} = Q_{1,5}Q_{1,6}\left(\lambda - 2 - \frac{1}{Q_{1,5}} - \frac{1}{Q_{1,6}}\right) = \lambda(\lambda - 1)(\lambda - 3).$$

For vertices on the second level of T we have

$$Q_{3,1} = \lambda - 1,$$

$$Q_{3,2} = Q_{2,1}Q_{2,2}\left(\lambda - 2 - \frac{1}{Q_{2,1}} - \frac{1}{Q_{2,2}}\right) = \lambda(\lambda - 1)(\lambda - 3),$$

$$Q_{3,3} = Q_{2,3}Q_{2,4}\left(\lambda - 2 - \frac{1}{Q_{2,3}} - \frac{Q_{1,1}}{Q_{2,4}}\right) = \lambda(\lambda - 2)(\lambda^2 - 4\lambda + 2),$$

$$Q_{3,4} = Q_{2,5}Q_{2,6}\left(\lambda - 2 - \frac{Q_{1,2}Q_{1,3}Q_{1,4}}{Q_{2,5}} - \frac{Q_{1,5}Q_{1,6}}{Q_{2,6}}\right)$$
$$= \lambda(\lambda - 1)^3(\lambda - 4)(\lambda^3 - 7\lambda^2 + 10\lambda - 2).$$

Therefore $Q_T(\lambda)$ can be computed as follows:

$$Q_{4,1} = \lambda^{17} - 32\lambda^{16} + 455\lambda^{15} - 3804\lambda^{14} + 20866\lambda^{13} - 79394\lambda^{12} + 216174\lambda^{11}$$
$$- 428560\lambda^{10} + 623610\lambda^{9} - 666420\lambda^{8} + 519469\lambda^{7} - 290878\lambda^{6} + 113966\lambda^{5}$$
$$- 29950\lambda^{4} + 4930\lambda^{3} - 450\lambda^{2} + 17\lambda.$$

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